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The Alfvén Invariant in the Field of a Magnetic Unipole

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It is well known $\underline{/}$Alfvén, 195$\overline{0}\underline{/}$ $\underline{/}$Northrop, 196$\overline{3}\underline{/}$ that an

isolated, electrically-charged particle moves in a static

magnetic field in such a way that the quantity

$$\frac{p_\perp^2}{2m_o B} \qquad (1)$$

is an adiabatic invariant of the motion provided that

$$\rho \left| \frac{\text{grad } B}{B} \right| << 1. \qquad (2)$$

In (1) and (2), $p_\perp$ is the component of the particle's

longitudinal, relativistic momentum $\underline{p}$ perpendicular to the

magnetic vector $\underline{B}$, m_o is its rest mass, and ρ is the radius

of curvature of the osculating circle to its trajectory.

Since $p_\perp = p \sin \alpha$, where α is the angle between $\underline{p}$ and $\underline{B}$

(instantaneous pitch angle), and since scalar p and m_o are

constants of the motion, it follows from the adiabatic invariance

of quantity (1) that

$$\frac{\sin^2 \alpha}{B} = \text{Adiabatic Constant.} \qquad (3)$$

The purpose of the present note is to call attention to the fact that $\sin^2 \alpha/B$ is a _rigorous_ constant of the motion of an isolated, electrically-charged particle in the field of a magnetic unipole.

It was shown by Darboux $\underline{/1878/}$ that the equilibrium form of a flexible, inextensible, current-carrying wire which is under tension and in the field of a magnetic unipole is a geodesic on a circular cone, whose apex is at the unipole. Later, Poincare $\underline{/1896/}$ pointed out that the differential equations of the trajectory of an electrically-charged particle are of identical form to those of Darboux's current-carrying wire provided one sets the ratio of the tension to the current equal to the magnetic rigidity of the particle. Hence, the trajectory of an electrically-charged particle is also a geodesic on a circular cone whose apex is at the unipole. The complete properties of the trajectory are determined by exact integration of the equations of motion with given initial conditions. A simple, elegant discussion using vector notation is given by Ferraro $\underline{/1956/}$.

It follows from the properties of geodesics on circular cones $\underline{/\text{Struik}, 1961/}$ that

$$r \sin \theta \sin \alpha = \text{constant} \tag{4}$$

where r is the radial distance from the apex of the cone to a point on the geodesic, θ is the half angle of the cone, and α is the angle between the geodesic line and the element of the cone through the point. For a unipole

$$B r^2 = \text{constant} \tag{5}$$

and $\underline{B}$ lies along the element of the cone.

From (4) and (5) it is seen that the Alfvén quantity $\sin^2 \alpha/B$ is a rigorous constant of the motion. Q.E.D.

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